Symbols:

$\exists$:= There exist

$\ni$:= Such that

$\subseteq$:= subset

$\supseteq$:= superset

$\cup$:= Union

$\cap$:= Intersection

Notations:

$\mathbb{N}$ = the set of all positive integers.

$= \{1, 2, 3, \ldots\}$

$\mathbb{Z}$ = the set of all integers

$= \{\ldots -2, -1, 0, 1, 2, \ldots\}$

$\mathbb{Q}$ = the set of all rational numbers

$= \{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0\}$.

$\mathbb{R}$ = the set of all real numbers.

$\mathbb{R} \setminus \mathbb{Q}$ = the set of all irrational numbers.

Note:-

1. $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.
2. $\mathbb{R} = (\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q}$
3. Every rational number can be written in the f m/n, where $m, n \in \mathbb{Z}$, ~~$n \neq 0$~~ $n > 0$ and $(m, n) = 1$.

Result:- $\sqrt{2}$ is not a ~~i~~rational number.

Pf: Suppose, on contrary, that $\sqrt{2} = m/n$ $\ni m, n$ $n > 0$ and $(m, n) = 1$.

Since $m^2 = 2n^2$, $\Rightarrow m^2$ is even $\Rightarrow m$ is e

(If m is odd, then $m = 2p - 1$, & $m^2 = (2p-1)^2$
$= 4p^2 - 4p + 1$

$\Rightarrow m^2 = 2(2p^2 - 2p + 1) - 1$ is also odd).

$\therefore$ Since m and n do not have 2 as a common factor, $(\because (m,n)=1)$

then n must be an odd natural number.

Since m is even, then $m = 2p$ for some $p \in \mathbb{N}$,

and hence $4p^2 = m^2 \Rightarrow 4p^2 = 2n^2 \Rightarrow 2p^2 = n^2$.

$\therefore$ n^2 is even $\Rightarrow$ n is even (same argument as above)

which is a contradiction as n cannot be both even and odd.

$\therefore$ $\sqrt{2}$ is not rational.

→ From the information we know, we can't say that $\sqrt{2}$ is irrational. For this, we need more information/properties of $\mathbb{R}$.

Now we discuss the essential properties of the real number system $\mathbb{R}$.

* Algebraic Properties of $\mathbb{R}$

On the set $\mathbb{R}$ of real numbers there are two binary operations, denoted by $+$ and $\cdot$ and called "addition" and "multiplication", respectively. These operations satisfy the following properties:

A1 : Commutative Property : $a+b = b+a \ \forall a, b \in \mathbb{R}$.

A2 : Associative Property : $a+(b+c) = (a+b)+c \ \forall a,b,c \in \mathbb{R}$

A3 : Additive identity : $\exists$ an element $0 \in \mathbb{R} \ni 0+a = a+0 = a$ $\forall a \in \mathbb{R}$.

A4 : Additive inverse : for each $a \in \mathbb{R}$ $\exists$ $-a \in \mathbb{R}$ $\ni$ $a+(-a) = 0$ and $(-a)+a = 0$.

M1: Commutative property w.r.t $\cdot$: $a \cdot b = b \cdot a \quad \forall a, b \in \mathbb{R}$

M2: Associative Property w.r.t $\cdot$: $a \cdot (b \cdot c) = (a \cdot b) \cdot c \quad \forall a, b, c \in \mathbb{R}$

M3: Multiplicative identity : $\exists$ an element $1 (\neq 0) \in \mathbb{R} \ni 1 \cdot a = a \cdot 1 = a \quad \forall a \in \mathbb{R}$.

M4: Multiplicative inverse: for each $a \neq 0$ in $\mathbb{R}$ $\exists$ an element $1/a$ in $\mathbb{R}$ $\ni$ $a \cdot (1/a) = 1$ and $(1/a) \cdot a = 1$

D: Distributive laws: For all $a, b, c \in \mathbb{R}$, we have

i, $a \cdot (b + c) = a \cdot b + a \cdot c$

ii, $(b + c) \cdot a = b \cdot a + c \cdot a$

* Order Properties of $\mathbb{R}$.

There is a non-empty subset $\mathbb{R}^+$ of $\mathbb{R}$, called the set of positive real numbers, that satisfies the following properties:

i, Given $a \in \mathbb{R}$, exactly one of the following holds: $a \in \mathbb{R}^+$ or $a = 0$ or $-a \in \mathbb{R}^+$.

ii, $a, b \in \mathbb{R}^+ \Rightarrow a + b \in \mathbb{R}^+$

iii, $a, b \in \mathbb{R}^+ \Rightarrow a \cdot b \in \mathbb{R}^+$.

Note:- Thanks to the existence of $\mathbb{R}^+$, we can now define the notion of inequality between two real numbers using $\mathbb{R}^+$.

Definition:- Let $a, b \in \mathbb{R}$.

We define i) $a < b$ if $b - a \in \mathbb{R}^+$.

ii) $a > b$ if $b < a$.

From the above, we can say $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ and

- $a, b \in \mathbb{R} \Rightarrow a < b$ or $a = b$ or $a > b$.
- $a, b \in \mathbb{R}$ & $a < b \Rightarrow a + c < b + c \;\; \forall\, c \in \mathbb{R}$

$ac < bc \;\; \forall\, c(>0) \in \mathbb{R}$

& $ac > bc \;\; \forall\, c(<0) \in \mathbb{R}$.

Examples:-

i) Let $a \geq 0$ and $b \geq 0$. Then

$$a \leq b \Leftrightarrow a^2 \leq b^2 \Leftrightarrow \sqrt{a} \leq \sqrt{b}.$$

ii) If $a, b \in \mathbb{R}^+$, then their "arithmetic mean" $\frac{1}{2}(a+b)$ is greater than or equal to their "geometric mean" $\sqrt{ab}$.

ie, $\sqrt{ab} \leq \frac{1}{2}(a+b)$.

iii) If $x > -1$, then $(1+x)^n \geq 1 + nx \;\; \forall\, n \in \mathbb{N}$.

This inequality is known as "Bernoulli's Inequality

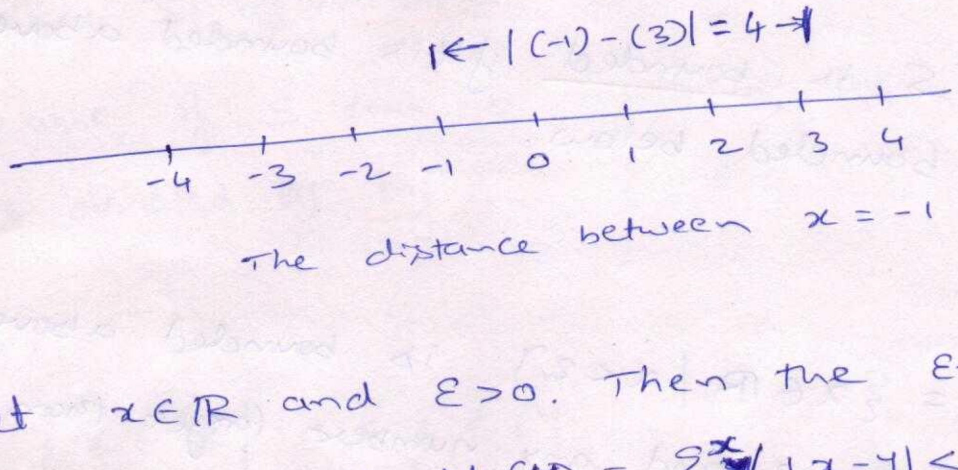
* Absolute value.

For $x \in \mathbb{R}$, $|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0. \end{cases}$

Basic facts: For any $x, y \in \mathbb{R}$,

- $|x+y| \le |x| + |y|$ (Triangle inequalit
- $\big| |x| - |y| \big| \le |x-y|$
- $|x-y| \le |x| + |y|$.

Note:- The absolute value $|x|$ of an element x in $\mathbb{R}$ is regarded as the distance from ~~x~~ to th origin 0. More generally, the distance between elements ~~x~~ and y in $\mathbb{R}$ is $|x-y|$.

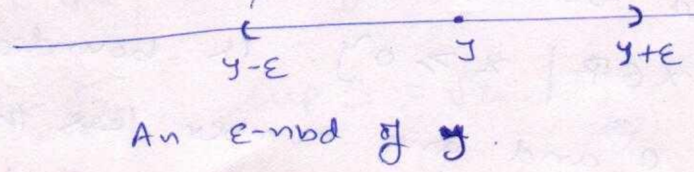


The distance between $x = -1$ and $y = 3$.

Def:- Let $x \in \mathbb{R}$ and $\varepsilon > 0$. Then the ε-neighborho of y is the set $N_\varepsilon(y) = \{x \,/\, |x-y| < \varepsilon\}$.

$$N_\varepsilon(x) \iff -\varepsilon < x - y < \varepsilon \iff y - \varepsilon < x < y + \varepsilon$$

ie,



An ε-nbd of y.

* Earlier we have seen that $\sqrt{2} \notin \mathbb{Q}$. This observation shows the necessity of an additional property to characterize the real number system.

This additional property, the completeness property, is an essential property of $\mathbb{R}$. To describe the completeness property, we first introduce the notions of upper bound & lower bound.

Def :- Let S be a non empty subset of $\mathbb{R}$.

- S is bounded above if $\exists\ u \in \mathbb{R} \ni$ $x \leq u \ \forall\ x \in S$.

  Any such u is called an upper bound of S.

- S is bounded below if $\exists\ w \in \mathbb{R} \ni$ $w \leq x \ \forall\ x \in S$.

  Any such w is called a lower bound of S.

- S is bounded if it is bounded above as well as bounded below.

Examples:

1. $S_1 = \{x \in \mathbb{R} \mid x < 2\}$ is bounded above. The number 2 and any number larger than 2 is an upper bound of S_1. Observe that $2 \notin S_1$.

   This set has no lower bounds, so this set is not bounded below.

2. $S_2 = \{x \in \mathbb{R} \mid x \geq 0\}$ is bounded below. The number 0 and any number less than 0 is ~~an~~ a lower bound of S_2. Observe that $0 \in S_2$

Definition: (Supremum (lub or sup) & Infimum (glb or Inf))

Let $S \subseteq \mathbb{R}$.

a) A real number M is called a supremum (sup) or a least upper bound (lub) of S if

- M is an u.b of S ie, $x \leq M \ \forall x \in S$.
- $M \leq u$ for any upper bound u of S.

b) Similarly, a real number m is called an Infimum (Inf) or a greatest lower bound (glb) of S if

- m is a l.b of S ie, $m \leq x \ \forall x \in S$.
- $w \leq m$ for any lowerbound w of S.

Uniqueness: It is easy to see that if a supremum exists, then it is unique. We denote the supremum of a set S by $\sup S$.

Likewise, if S has an infimum, then it is unique and is denoted by $\inf S$.

Examples:-

1. $S = \{x \in \mathbb{R} \mid 0 < x \leq 1\}$

$\inf S = 0$, $\sup S = 1 = \max S \ (\because 1 \in S)$.

2. $S = \{x \in \mathbb{Q} \mid x^2 < 2\}$

$\inf S = -\sqrt{2}$, $\sup S = \sqrt{2}$.

Note:- $S \subseteq \mathbb{Q}$ but sup & inf S neither belongs S

* Completeness Property of $\mathbb{R}$:

Every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum in $\mathbb{R}$.

With this property, we can conclude that $\sqrt{2}$ is an element of $\mathbb{R}$ and is not a rational number. Hence $\sqrt{2}$ is an irrational number.

Note:- Using Completeness Property of $\mathbb{R}$, we can also conclude the following: (along with algebraic & order properties too)

1. "Every non-empty subset of $\mathbb{R}$ that is bounded below has an infimum in $\mathbb{R}$".

2. [Archimedian Property]

   Given $x \in \mathbb{R}$, $\exists\, n \in \mathbb{N} \ni n > x$.

3. [Existence of n^{th} roots]

   Given $n \in \mathbb{N}$ and $a \in \mathbb{R}$ with $a \geq 0$, $\exists$ unique $b \in \mathbb{R} \ni b \geq 0$ and $b^n = a$.

   Here $b = a^{1/n}$ or $b = \sqrt[n]{a}$.

4. [Density Theorem]

   - Given any $a, b \in \mathbb{R}$ with $a < b$, there is a rational as well as irrational between a & b.

   - Between any two real numbers, there are infinitely many rationals as well as irrationals.

Intervals:

If $a, b \in \mathbb{R}$ satisfy $a < b$, then

- open interval, $(a,b) := \{x \in \mathbb{R} \mid a < x < b\}$
- Closed interval, $[a,b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}$.
- Semi-open intervals, (or semi-closed)

$$(a, b] := \{x \in \mathbb{R} \mid a < x \leq b\}$$

& $$[a, b) := \{x \in \mathbb{R} \mid a \leq x < b\}.$$

There are five types of unbounded intervals for which the symbols ∞ and $-\infty$ are used as notation convenience. They are:

i, $(a, \infty) := \{x \in \mathbb{R} \mid x > a\}$

ii, $(-\infty, b) := \{x \in \mathbb{R} \mid x < b\}$.

Infinite open-intervals.

iii, $[a, \infty) := \{x \in \mathbb{R} \mid x \geq a\}$

iv, $(-\infty, b] := \{x \in \mathbb{R} \mid x \leq b\}$

Infinite closed intervals

v, $(-\infty, \infty) := \mathbb{R}$.

General Definition: A subset I of $\mathbb{R}$ is called an interval if $x, y \in I$, $x < y \Rightarrow [x, y] \subseteq I$.

Sequences

Definition:

A sequence in $\mathbb{R}$ is a function from $\mathbb{N}$ to $\mathbb{R}$.

* If $X : \mathbb{N} \to \mathbb{R}$ is a sequence, we will usually denote the value of X at n by the symbol x_n rather than using the function notation $X(n)$.

we usually denote a sequence by $\langle x_n \rangle$.

x_n is called the n^{th} term of the seq. $\langle x_n \rangle$.

Examples:

- $\langle n \rangle \sim 1, 2, 3, \ldots$
- $\langle (-1)^n \rangle \sim -1, 1, -1, 1, -1, 1, \ldots$
- $\langle \frac{1}{2^n} \rangle \sim \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$
- $x_1 = 1$, $x_2 = 1$, $x_{n+1} = x_{n-1} + x_n$ $(n \geq 2)$.

  Then $\langle x_n \rangle = 1, 1, 2, 3, 5, 8, \ldots$

  * This sequence is known as "Fibonacci Sequence".

Basic Definitions:

Def: A sequence $\langle x_n \rangle$ is said to be

- bounded above if the set $\{x_1, x_2, \ldots\}$ of its terms is bounded above, i.e., $\exists\, \alpha \in \mathbb{R}$ s.t. $x_n \leq \alpha$ $\forall n \in \mathbb{N}$.
- bounded below if the set $\{x_1, x_2, \ldots\}$ of its terms is bounded below, i.e., $\exists\, \beta \in \mathbb{R}$ s.t $x_n \geq \beta$ $\forall n \in \mathbb{N}$
- bounded if $\exists\, M (> 0) \in \mathbb{R}$ $\ni$ $|x_n| \leq M$ $\forall n \in \mathbb{N}$.

Examples:-

- $\langle(-1)^n\rangle$, $\langle\frac{1}{n}\rangle$, $\langle\frac{1}{2^n}\rangle$, $\langle\frac{1}{2n+1}\rangle$ are bounded,
- $\langle n\rangle$, $\langle 2^n\rangle$, Febonacci sequence are bounded below but not bounded above.

Def:- A sequence $\langle x_n\rangle$ is

- increasing (or monotonically increasing) if $x_1 \le x_2 \le x_3 \le \cdots$, i.e, $x_n \le x_{n+1}$ $\forall n \in \mathbb{N}$.
- decreasing (or monotonically decreasing) if $x_1 \ge x_2 \ge x_3 \ge \cdots$, i.e, $x_n \ge x_{n+1}$ $\forall n \in \mathbb{N}$.
- Monotonic if it is either increasing or decreasing.

Examples:

- $\langle 1, 2, 3, 4, \ldots, n, \ldots\rangle = \langle n\rangle$,

  $\langle a^n\rangle$, where $a > 1$,

  $\langle 2^n\rangle$, fibonacci sequence

  are increasing sequences.
- $\langle\frac{1}{n}\rangle$, $\langle\frac{1}{2^n}\rangle$, $\langle b^n\rangle$, where $0 < b < 1$, are decreasing sequences.
- $\langle(-1)^n\rangle$, $\langle(-1)^n n\rangle$ are not monotonic.

Most important def. (Convergence / limit of a sequence)

Def:- A sequence $\langle x_n\rangle$ is said to be Convergent if $\exists$ a real number l such that for every $\varepsilon > 0$,

$\exists\, n_0 \in \mathbb{N}$

$$|x_n - l| < \varepsilon \quad \forall\, n \geq n_0.$$

Intuitively this means that the terms approach l as n becomes larger and larger.

Note: If a sequence has a limit, we say that the sequence is convergent; A sequence is said to be divergent if it is not convergent.

Observations:

1. A sequence in $\mathbb{R}$ can have at most one limit. ie., if the limit exists, then that limit is unique.
2. If $\langle x_n \rangle$ cgs to l, then, we write $x_n \to l$ to indicate $\lim\limits_{n\to\infty} x_n = l$.
3. The convergence of a sequence is unaltered if a finite number of its terms are replaced by other terms.
4. $\lim\limits_{n\to\infty} x_n = l \iff$ for each ε-nbd $N_\varepsilon(l)$, all but a finite number of terms of $\langle x_n \rangle$ belongs to $N_\varepsilon(l)$.

Examples:

1. If $\langle x_n \rangle$ is a constant sequence, ie, $x_n = c\ \forall n \in$ then $x_n \to c$.

Here for a given $\varepsilon > 0$, we can choose any $n_0 \in \mathbb{N}$ which satisfies the condition $|x_n - c| < \varepsilon \ \forall\, n \geq n_0$.

2. If $x_n = \frac{1}{n}$ $\forall\, n \in \mathbb{N}$, then $x_n \to 0$.

ie, $\lim_{n\to\infty} x_n = 0$.

Sol: If $\varepsilon > 0$ is given, then $\frac{1}{\varepsilon} > 0$.

By the Archimedean Property, $\exists\, K \in \mathbb{N} \ni$

$\frac{1}{K} < \varepsilon$.

Then, if $n \geq K$, we have $\frac{1}{n} \leq \frac{1}{K} < \varepsilon$.

$\therefore$ Given $\varepsilon > 0$, choose $K \in \mathbb{N} \ni \frac{1}{K} < \varepsilon$, then

$$\left|\frac{1}{n} - 0\right| \leq \frac{1}{n} < \varepsilon \quad \forall\, n \geq K.$$

$\therefore \frac{1}{n} \to 0$.

3. The sequence $\langle (-1)^n \rangle$ is not convergent.

Sol: Suppose if $\exists\, l \in \mathbb{R} \ni (-1)^n \to l$, then for $\varepsilon := 1$

we could find $n_0 \in \mathbb{N} \ni |(-1)^n - l| < \varepsilon$ $\forall\, n \geq n_0$.

$$\Rightarrow |(-1)^n - l| < 1 \quad \forall\, n \geq n_0.$$

In particular $|(-1)^{2n_0} - l| < 1$ & $|(-1)^{2n_0+1} - l| < 1$

$$\therefore 2 = |(-1)^{2n_0} - (-1)^{2n_0+1}| \leq |(-1)^{2n_0} - l| + |(-1)^{2n_0+1} - l|$$

$$< 1 + 1 = 2$$

$\Rightarrow 2 < 2$, which is a contradiction.

$\therefore \langle (-1)^n \rangle$ is not convergent.

Some basic Results:

1. Every convergent sequence of realnumbers is bounded.

Proof: Suppose that $x_n \to l$ and let $\varepsilon = 1$.

Then $\exists\, n_0 \in \mathbb{N} \ni |x_n - l| < 1 \quad \forall\, n \geq n_0$.

Now $|x_n| = |x_n - l + l| \leq |x_n - l| + |l|$

$< 1 + |l| . \ \forall\, n \geq n_0$.

If we set $M := \max\{|x_1|, |x_2|, \ldots, |x_{n_0}|, 1+|l|\}$

then $|x_n| \leq M \quad \forall\, n \in \mathbb{N}$.

* As a consequence, we can immediately say that $\langle n \rangle$ and $\langle 2^n \rangle$ are divergent. (as these sequences are unbounded).

2. Let $\langle x_n \rangle$ and $\langle y_n \rangle$ be sequences of real no's that converge to l and m, respectively, and let $c \in \mathbb{R}$. Then
   - $x_n + y_n \to l + m$
   - $c x_n \to cl$
   - $x_n \cdot y_n \to lm$
   - If $m \neq 0$, then $\exists\, N \in \mathbb{N} \ni y_n \neq 0 \quad \forall\, n \geq N$ and the seq. $\left\langle \frac{x_n}{y_n} \right\rangle_{n \geq N}$ converges to $\frac{l}{m}$.

Proof: [Take it as an excercise]

3. If $x_n \to l$ then $|x_n| \to |l|$. Converse neednot

4. If $\langle x_n \rangle$ is a convergent sequence of real no.'s and if $x_n \geq 0 \ \forall n \in \mathbb{N}$, then $\lim_{n\to\infty} x_n = x \geq 0$.

Proof:- On the contrary, if $x < 0$; then for $\varepsilon = -x > 0$,
$\exists n_0 \in \mathbb{N} \ni x - \varepsilon < x_n < x + \varepsilon \quad \forall n \geq n_0$.

In particular, $0 \leq x_N < x + \varepsilon = x + (-x) = 0$.

$\therefore 0 \leq x_N < 0$, which contradicts.

$\therefore x \geq 0$.

5. [Sandwich / squeeze Theorem]

Suppose $x_n \leq y_n \leq z_n \ \forall n \in \mathbb{N}$, and suppose that $\lim x_n = \lim z_n = l$. Then $\lim y_n = l$.

Proof:- for $\varepsilon > 0 \ \exists n_0 \in \mathbb{N} \ni |x_n - l| < \varepsilon$ & $|z_n - l| < \varepsilon \quad \forall n \geq n_0$.

$\because x_n - l \leq y_n - l \leq z_n - l \ \forall n \in \mathbb{N}$ [we can choose such ~~large~~ no which satisfy both.]

$\Rightarrow -\varepsilon < x_n - l \leq y_n - l \leq z_n - l < \varepsilon. \quad \forall n \geq n_0$.

$\therefore y_n \to l$.

Examples:-

1. $x_n = \dfrac{\sin n}{n}$

Note that $-1 \leq \sin n \leq 1$.

Then it follows that

$$\frac{-1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n} \quad \forall n \in \mathbb{N}.$$

$\because -1/n \to 0$ & $\frac{1}{n} \to 0$, by Sandwich / squeeze th.,

$\frac{\sin n}{n} \to 0$. $\therefore \lim \left(\frac{\sin n}{n}\right) = 0$.

2. Let $x \in \mathbb{R}$ with $|x|<1$. Define $x_n = x^n$ for $n \in$

If $x=0$, then clearly $x_n \to 0$.

Suppose $x \neq 0$. Then $\frac{1}{|x|} > 1$ and hence we can write $\frac{1}{|x|} = 1+K$ with $K>0$.

$$\Rightarrow \frac{1}{|x|^n} = (1+K)^n = 1+nK+\binom{n}{2}K^2+\cdots \geq nK \quad \forall n \in \mathbb{N}.$$

$$\therefore \quad 0 < |x_n| = |x|^n \leq \frac{1}{nK} \to 0 \text{ as } n \to \infty.$$

$$\Rightarrow x_n \to 0.$$

*

Theorem:

A monotone sequence of real numbers is convergent if and only if it is bounded.

More precisely,

a) If $\langle x_n \rangle$ is a bounded (bounded above is enough to be precise) increasing sequence, then $\lim x_n = \sup\{x_n / n \in \mathbb{N}\}$.

b) If $\langle y_n \rangle$ is a bounded (bounded below is enough) decreasing seq., then $\langle y_n \rangle$ is convergent and $y_n \to \inf\{y_n \mid n \in \mathbb{N}\}$.

Proof: a) Suppose $\langle x_n \rangle$ is increasing and bounded above. Then $M = \sup\{x_1, x_2, \ldots\}$ exists.

Given any $\varepsilon > 0$, we have $M-\varepsilon < M$ and so $\exists\, n_0 \in \mathbb{N} \ni x_{n_0} > M-\varepsilon$.

But then $M-\varepsilon < x_{n_0} \leq x_n \leq M < M+\varepsilon \quad \forall n \geq n_0$

$\Rightarrow |x_n - M| < \varepsilon \ \forall \ n \geq n_0.$

$\therefore x_n \to M.$

b) [Proof is similar as in (a)].

<u>Example</u> :-

Consider $x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$

Clearly $\langle x_n \rangle$ is an increasing sequence.

Also $x_n \leq 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} = 1 + 2\left(1 - \frac{1}{2^n}\right)$

$< 3.$

$\therefore$ By monotone cgs. th, $\langle x_n \rangle$ is convergent.

*